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ELEMENTS OF MAGNETOHYDRODYNAMIC STABILITY THEORY

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In times when scientists are expected to compute answers no matter whether the questions are understood, I feel obliged to thank Professor Harold Grad for providing an athmosphere in which it is still possible to reflect about basic theory.

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Thanks are due to Professors Thomas Dupree and Dieter Sigmar of the Massachusetts Institute of Technology for inviting me to present these lectures in their class.

Preface

These notes contain the material presented during a series of lectures at the Massachusetts Institute of Technology in May, 1975. It was the intent of these lectures to familiarize the students with the basic concepts of MHD stability theory and to lead them to a point where they are able to understand and interpret the literature on this subject. Therefore, only a rather limited number of applications is given which are by no means representative for the present status of the theory.

Due to its relative simplicity, MHD theory can be used to study the effect of the geometry of the magnetic field upon plasma confinement. Since this question is crucial for the problem of Controlled Thermonuclear Fusion, MHD is one of the most extensively studied subjects in plasma theory, and there exists a vast literature. Nevertheless, it should be emphasized that MHD stability is not very well understood at present. For one thing, it is not clear how one should define stability in order to obtain a problem which is both physically relevant and mathematically tractable. Moreover, it has not been possible to gain a generally valid qualitative picture within any of the common definitions of stability. Correspondingly, it is not exaggerating to say that the relevant questions cannot be answered without

extensive numerical computations. The practically important knowledge thus consists largely of more or less unrelated special cases which, nevertheless, are extremely involved. Naturally, none of these can be presented here.

Due to the somewhat unusual point of view taken in these lectures, much of the material included has been gathered from relatively inaccessible or even unpublished sources. Therefore, and consistently with the attempt to keep these notes self-contained, a list of references is not included. It is hoped that the numerous original authors accept this apology.

1. Some Simple Consequences of the Nonlinear Equations

We write the nonlinear equations of ideal MHD in the form

$$\rho\frac{d\underset{\sim}{u}}{dt} = -\nabla p + \underset{\sim}{J} \times \underset{\sim}{B} \qquad (1.1)$$

$$\underset{\sim}{J} = \text{curl}\ \underset{\sim}{B} \qquad (1.2)$$

$$\text{div}\ \underset{\sim}{B} = 0 \qquad (1.3)$$

$$\frac{\partial\underset{\sim}{B}}{\partial t} = \text{curl}\ (\underset{\sim}{u} \times \underset{\sim}{B}) \qquad (1.4)$$

$$\frac{\partial\rho}{\partial t} + \text{div}\ \rho\underset{\sim}{u} = 0 \qquad (1.5)$$

$$\frac{d}{dt}\ \frac{p}{\rho^{\gamma}} = 0 \qquad (1.6)$$

Here $\partial/\partial t$ is the time derivative at a fixed position, and d/dt is the time derivative following the fluid motion:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{u}\cdot\nabla \qquad (1.7)$$

These equations are not sufficiently tractable to serve as a basis for a stability theory. Therefore, we consider their linearized versions in most of what follows. However, we derive a few general consequences which help us to develop some intuition.

Conservation of Energy

The equations (1.1-7) imply the identity

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\, u^2 + \frac{1}{\gamma-1}p + \frac{1}{2}\, B^2\right) \tag{1.8}$$

$$+ \operatorname{div}\left[\left(\frac{1}{2}\rho\, u^2 + \frac{\gamma}{\gamma-1}p\right)\underset{\sim}{u} + \underset{\sim}{B} \times (\underset{\sim}{u}\times\underset{\sim}{B})\right] = 0$$

Integrating this over a fixed spatial domain, we can use Gauss' theorem to show that the second term does not contribute if we adopt the boundary conditions

$$u_n = 0 \quad , \quad B_n = 0 \tag{1.9}$$

(which are appropriate for a rigid and perfectly conducting wall). A subscript n denotes the normal component of the vector. We thus have

$$\frac{d}{dt}\iiint d\tau,\left(\frac{1}{2}\rho u^2 + \frac{1}{\gamma-1}p + \frac{1}{2}\, B^2\right) = 0 \tag{1.10}$$

The three terms in the integrand are the kinetic, thermal, and magnetic energy densities. The integral of the first is the kinetic energy (denoted by K), and the integral of the sum of the of the last two is the potential energy (denoted by U).

Frozen-in Magnetic Field Lines

We show that magnetic field lines can be viewed as moving with the fluid velocity $\underset{\sim}{u}$. We use the fact that equation (1.3) is equivalent to

$$\underset{\sim}{B} = \nabla\alpha \times \nabla\beta. \tag{1.11}$$

It is trivially seen that equation (1.11) implies equation (1.3), but to show the converse, viz., that any field satisfying equation (1.3) can be represented by two "flux functions" α and β according to equation (1.11), requires some thought. Equation (1.11) implies that both α and β are constant along field lines. Hence each field line can be viewed as labeled by a pair of values of α and β. If α and β depend on time, each field line moves accordingly. To prove the assertion we show that equation (1.11) implies equation (1.4) if we let α and β depend on time according to

$$\frac{d\alpha}{dt} = 0 \quad , \quad \frac{d\beta}{dt} = 0. \tag{1.12}$$

We have

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \frac{\partial}{\partial t}(\nabla\alpha\times\nabla\beta) = \nabla\frac{\partial\alpha}{\partial t} \times \nabla\beta+\nabla\alpha\times\nabla\frac{\partial\beta}{\partial t}$$

$$= -\nabla(\underset{\sim}{u}\cdot\nabla\alpha)\times\nabla\beta-\nabla\alpha\times\nabla(\underset{\sim}{u}\cdot\nabla\beta).$$

On the other hand

$$\begin{aligned}
\mathrm{curl}(\underset{\sim}{u}\times\underset{\sim}{B}) &= \mathrm{curl}[(\underset{\sim}{u}\times(\nabla\alpha\times\nabla\beta)]\\
&= \mathrm{curl}[(\underset{\sim}{u}\cdot\nabla\beta)\nabla\alpha-(\underset{\sim}{u}\cdot\nabla\alpha)\nabla\beta]\\
&= \nabla(\underset{\sim}{u}\cdot\nabla\beta)\times\nabla\alpha-\nabla(\underset{\sim}{u}\cdot\nabla\alpha)\times\nabla\beta.
\end{aligned}$$

Combining of the last two relations yields equation (1.4).

The statement that field lines move with the fluid velocity is not an absolute one because there is a whole class of velocity fields $\underset{\sim}{\hat{u}}$, characterized by

$$\mathrm{curl}[(\underset{\sim}{\hat{u}} - \underset{\sim}{u})\times\underset{\sim}{B} = 0,$$

which can be viewed as field line velocities. The point is that $\underset{\sim}{u}$ is a possible field line velocity. The sifnificance of this is that an identity can be assigned to each field line for all times. Therefore, field lines can bever break or combine. This is often referred to as conservation of the topology of field lines. For instance, a closed field line will be closed forever, two closed field lines will be linked in the same way forever, and field lines which lie in some surface will lie in a similar surface forever.

Circulation Theorem

The circulation along some closed path is defined as

$$C = \oint d\underset{\sim}{\ell} \cdot \underset{\sim}{u} \tag{1.12}$$

In isentropic ideal hydrodynamics C is conserved for any closed path if this path is carried with the fluid motion. In isentropic ideal MHD, C is conserved for paths which are formed by magnetic field lines. We call the plasma isentropic if $p = \eta\rho^{\gamma}$ with constant η. The surfaces of constant pressure then coincide with those of constant density.

To prove the circulation theorem of MHD, let us first derive an expression for the time derivative of an integral of the form $\oint d\underset{\sim}{\ell} \cdot \underset{\sim}{A}$, where $\underset{\sim}{A}$ is any time dependent vector field, and the integral is along a closed magnetic field line.

It is convenient to use the flux functions α and β, supplemented by a third function χ subject to $d\chi/dt = 0$, as coordinates. Then

$$d\underset{\sim}{\ell} = d\chi\, \underset{\sim}{B}/J, \tag{1.13}$$

where

$$J = (\nabla\alpha\times\nabla\beta)\nabla\chi = \underset{\sim}{B}\cdot\nabla\chi,$$

Since these coordinates are moving with the fluid, the operator d/dt equals the time derivative at fixed α,β, and χ, and thus commutes with the χ-integration. Hence

$$\frac{d}{dt}\oint d\underset{\sim}{\ell}\cdot\underset{\sim}{A} = \frac{d}{dt}\int d\chi\,\frac{\underset{\sim}{A}\cdot\underset{\sim}{B}}{J} = \int d\chi\frac{d}{dt}\,\frac{\underset{\sim}{A}\cdot\underset{\sim}{B}}{J}$$

$$= \int \frac{d\chi}{J}\left(\frac{d}{dt}(\underset{\sim}{A}\cdot\underset{\sim}{B}) - \frac{\underset{\sim}{A}\cdot\underset{\sim}{B}}{J}\,\frac{dJ}{dt}\right)$$

Using equations (1.3-4) and the relation

$$\frac{dJ}{dt} = -J \text{ div } \underset{\sim}{u} \tag{1.14}$$

(which can be verified by a straightforward calculation) we find

$$\frac{d}{dt}(\underset{\sim}{A}\cdot\underset{\sim}{B}) - \frac{\underset{\sim}{A}\cdot\underset{\sim}{B}}{J}\,\frac{dJ}{dt}$$

$$= \underset{\sim}{B}\cdot\left(\frac{\partial\underset{\sim}{A}}{dt} - \underset{\sim}{u}\times\text{curl } \underset{\sim}{A} + \nabla(\underset{\sim}{A}\cdot\underset{\sim}{u})\right)$$

Now, since the last term integrates to zero,

$$\frac{d}{dt}\oint d\underset{\sim}{\ell}\cdot\underset{\sim}{A} = \oint d\underset{\sim}{\ell}\cdot\left(\frac{\partial \underset{\sim}{A}}{\partial t} - \underset{\sim}{u}\times \text{curl } \underset{\sim}{A}\right) \qquad (1.15)$$

For the circulation theorem, we put $\underset{\sim}{A} = \underset{\sim}{u}$ to obtain

$$\frac{dC}{dt} = \oint d\underset{\sim}{\ell}\cdot\left(\frac{\partial \underset{\sim}{u}}{\partial t} - \underset{\sim}{u}\times \text{curl } \underset{\sim}{u}\right)$$

Using Eq. (1.1), the integrand can be written as

$$\frac{\partial \underset{\sim}{u}}{\partial t} - \underset{\sim}{u}\times\text{curl } \underset{\sim}{u} = -\nabla\left(\frac{n}{\gamma+1}\rho^{\gamma+1} + \frac{1}{2}u^2\right) + \frac{1}{\rho}\underset{\sim}{J}\times\underset{\sim}{B}$$

The first term then does not contribute to the integral because it is a gradient, and the second term does not contribute because it is perpendicular to the path of integration.

Well-posedness

Before we go on to the linear theory, we would like to stress one important point: the nonlinear MHD equations are symmetric hyperbolic, and therefore the initial value problem is well-posed. More precisely, for any initial data which are compatible with the equations, and with the boundary conditions (1.9), the equations possess exactly one solution which depends continuously on the initial data. Even though we do not work with the nonlinear equations, we keep this fact in mind whenever we encounter difficulties with the equilibrium equations or with the linearized equations: one might sometimes be tempted to

conclude that the MHD model is not good if such difficulties occur; the point which we are making here is that the MHD model is, at least from the mathematical point of view, as good as a model can be, and difficulties arise only if we ask unreasonable questions. It is reasonable to require that the temporal evolution of the system is uniquely (and continuously) determined by its initial state; but there is no logical necessity for the system to relax to an equilibrium state, nor for equilibrium states to exist at all.

2. Static Equilibrium

An equilibrium is a time independent solution of the equations of motion. For simplicity, we restrict ourselves to static equilibria, i.e. equilibria with $\underset{\sim}{u} = 0$. The equations for these are

$$\nabla p = \underset{\sim}{J} \times \underset{\sim}{B} \tag{2.1}$$

$$\underset{\sim}{J} = \text{curl } \underset{\sim}{B} \tag{2.2}$$

$$\text{div } \underset{\sim}{B} = 0 \tag{2.3}$$

An immediate consequence of equation (2.1) is

$$\underset{\sim}{B} \cdot \nabla p = 0 \tag{2.4}$$

This means that the magnetic field lines lie in pressure surfaces, which thus are "magnetic surfaces". For confined equilibria, these surfaces must be closed. It can be shown that equation (2·3) is not compatible with closed magnetic surfaces unless each surface encloses a multiply connected domain, such as a torus. This is why one is primarily interested in toroidal equilibria. In a toroidal equilibrium the pressure surfaces form nested sets of toroids. In a confined equilibrium the outermost surface carries the lowest pressure. It is convenient to assume that this outermost surface coincides with a

perfectly conducting rigid well. The magnetic field lines (and also the field lines of the current density) lie in the toroidal surfaces.

It can be shown that a "rotation number" can be assigned to each magnetic field line. This is the limit of the ratio of the number of windings the short and the long way around of a magnetic field line as one follows it forever. The rotation number is the same for all field lines in the same surface. The field lines in a particular surface are closed if the rotation number is rational there, and each field line covers the surface densely otherwise. For instance, the field lines close upon themselves after one circuit the long way around without encircling the short way if the rotation number is zero; but they close the short way around without encircling the long way if the rotation number is infinite. A surface with rational rotation number is called "rational surface." If the rotation number varies from one surface to another, the equilibrium is said to have "shear". The rational surfaces are then dense, but almost all surfaces are "irrational". A special class of shearless equilibria (the rotation number is constant) are those with all field lines closed (the rotation number is a rational constant). Closed line equilibria are special in many respects, and their theory is particularly well developed.

The equilibrium equations (2.1-3) are neither elliptic nor hyperbolic: the magnetic field lines are (degenerate) characteristics, but there is also an imaginary cone. As

a consequence, the theory of the equilibrium equations (2.1-3) is by no means understood. It has been possible to formulate well posed boundary value problems for simply connected domains (the real characteristics intersect the boundary), for toroidal domains with axial symmetry (the hyperbolic part of the equations can be integrated, so that the problem reduces to a purely elliptic one), and for certain equilibria with zero rotation number (the real characteristics are closed curves). However, it seems likely that it is not possible to formulate a well posed boundary value problem for arbitrary toroidal domains because of the complicated topological structure of the real characteristics (dense set of rational surfaces). In other words, it seems likely that an equilibrium does either not exist for arbitrary domains, or, if it happens to exist, does not depend continuously on the parameters describing the shape of the domain. In the present notes we will not be concerned with equilibrium theory, but rather adopt the point of view that some equilibrium is given, and then investigate motions near this equilibrium.

3. Strict Linear Stability

We say that an equilibrium is "stable" if the plasma, according to the nonlinear equations of motion, stays near this equilibrium forever if it was near initially. Unfortunately, this definition is neither unique nor useful. It is not unique because there are many ways of defining what one means by "near". For instance, one could require that the deviations from their equilibrium values of all quantities appearing in the equations of motion are small everywhere, or one could require that the kinetic energy is small. The above definition is not useful because we are not able to handle the nonlinear equations well enough to decide whether a given equilibrium is stable or not.

A major simplification consists in linearizing the equations about the equilibrium. One writes $\underset{\sim}{B} = \underset{\sim}{B}_o + \underset{\sim}{B}_1$, $\underset{\sim}{u} = \underset{\sim}{u}_1$, etc., where quantities with a subscript zero satisfy the equilibrium equations, and then neglects terms which are quadratic in the perturbing quantities. The equations (1.1-6) then become

$$\rho_o \frac{\partial \underset{\sim}{u}_1}{\partial t} = -\nabla p_1 + \underset{\sim}{J}_1 \times \underset{\sim}{B}_o + \underset{\sim}{J}_o \times \underset{\sim}{B}_1 \ , \tag{3.1}$$

$$\underset{\sim}{J}_1 = \text{curl } \underset{\sim}{B}_1 \ , \tag{3.2}$$

$$\text{div } \underset{\sim}{B}_1 = 0 \ , \tag{3.3}$$

$$\frac{\partial \underset{\sim}{B}_1}{\partial t} = \text{curl}(\underset{\sim}{u}_1 \times \underset{\sim}{B}_o) \ , \tag{3.4}$$

$$\frac{\partial \rho_1}{\partial t} + \text{div}(\rho_o \underset{\sim}{u}_1) = 0 \ , \tag{3.5}$$

$$\frac{\partial p_1}{\partial t} = -\underset{\sim}{u}_1 \cdot \nabla p_o - \gamma p_o \text{ div } \underset{\sim}{u}_1 \ . \tag{3.6}$$

This is a set of linear homogeneous equations for the perturbing quantities. Like their nonlinear counterparts, these equations define a well-posed initial value problem. Nevertheless, it is unknown to what extent the solutions of the nonlinear equations are approximated by solutions of the linearized equations.

Since the linearized equations are homogeneous, smallness of a solution is equivalent to boundedness. We thus call an equilibrium stable if all solutions of the linearized equations remain bounded as $t\to\infty$. Of course, we have to decide what we mean by "bounded". In other words, we have to introduce some norm in the space of vector functions $(\underset{\sim}{B}_1, \underset{\sim}{u}_1, \ldots\ldots)$, and then call a solution bounded if this norm is finite. One possibility is the maximum norm. We then call the equilibrium stable if the quantities $\underset{\sim}{B}_1, \underset{\sim}{u}_1, \ldots.$ remain everywhere bounded as $t\to\infty$ for all solutions of equations(3.1-6). To distinguish this definition from others, we call it "strict linear stability." This definition might seem reasonable at first glance, but unfortunately virtually no equilibrium of interest is stable according to it.

To show this, we pick some closed field line, and define the first order circulation as

$$c(t) = \oint d\underset{\sim}{\ell} \cdot \underset{\sim}{u}_1 \quad . \qquad (3.7)$$

Since the equilibrium is static, the path of integration does not move to lowest order, and the time derivative of

c is obtained by just differentiating the integrand. Using the linearized equations of motion and the equilibrium equations (and again assuming that $\nabla p_o \times \nabla \rho_o = 0$), we compute

$$\begin{aligned}
\frac{dc}{dt} &= \oint d\underset{\sim}{\ell}\cdot\frac{\partial \underset{\sim}{u}_1}{\partial t} \\
&= \frac{1}{\rho_o}\oint d\underset{\sim}{\ell}\cdot(-\nabla p_1+\underset{\sim}{J}_1\times\underset{\sim}{B}_o+\underset{\sim}{J}_o\times\underset{\sim}{B}_1) \\
&= \frac{1}{\rho_o}\oint d\underset{\sim}{\ell}\cdot(\underset{\sim}{J}_o\times\underset{\sim}{B}_1) \\
&= \frac{1}{\rho_o}\oint d\ell\ \frac{1}{B_o}\ \underset{\sim}{B}_o\cdot(\underset{\sim}{J}_o\times\underset{\sim}{B}_1) \\
&= -\frac{1}{\rho_o}\oint d\ell\ \frac{1}{B_o}\ \underset{\sim}{B}_1\cdot(\underset{\sim}{J}_o\times\underset{\sim}{B}_o) \\
&= -\frac{1}{\rho_o}\oint d\ell\ \frac{1}{B_o}\ (\underset{\sim}{B}_1\cdot\nabla p_o), \qquad (3.8)
\end{aligned}$$

and further

$$\begin{aligned}
\frac{d^2c}{dt^2} &= -\frac{1}{\rho_o}\oint d\ell\ \frac{1}{B_o}\ (\frac{\partial \underset{\sim}{B}_1}{\partial t}\cdot\nabla p_o) \\
&= -\frac{1}{\rho_o}\oint d\ell\ \frac{1}{B_o}\ \nabla p_o\cdot\mathrm{curl}(\underset{\sim}{u}_1\times\underset{\sim}{B}_o) \\
&= -\frac{1}{\rho_o}\oint d\ell\ \frac{1}{B_o}\ \mathrm{div}((\underset{\sim}{u}_1\times\underset{\sim}{B}_o)\times\nabla p_o) \\
&= -\frac{1}{\rho_o}\oint d\ell\ \frac{1}{B_o}\ \mathrm{div}(\underset{\sim}{u}_1\cdot\nabla p_o)\underset{\sim}{B}_o) \\
&= -\frac{1}{\rho_o}\oint d\underset{\sim}{\ell}\ \frac{1}{B_o}\ \underset{\sim}{B}_o\cdot\nabla(\underset{\sim}{u}_1\cdot\nabla p_o) \\
&= -\frac{1}{\rho_o}\oint d\underset{\sim}{\ell}\cdot\nabla(\underset{\sim}{u}_1\cdot\nabla p_o) = 0 \qquad (3.9)
\end{aligned}$$

Hence dc/dt is constant, and

$$c(t) = c(t_o)+\frac{dc}{dt}(t-t_o) \quad . \tag{3.10}$$

The initial data can be chosen such that $dc/dt \neq 0$ unless $\nabla p_o = 0$ at the field line under consideration. Then $|c|\to\infty$ and hence also $\max|\underset{\sim}{u}_1|\to\infty$ as $t\to\infty$, and the equilibrium is unstable. The only equilibria which cannot be shown to be unstable in this manner are those which either have no closed field lines at all or in which the pressure gradient is zero at each closed field line (which, for reasons of continuity, requires that the pressure is constant wherever the rotation number is not an irrational constant).

It is interesting to note that $dc/dt = 0$ for those perturbing magnetic fields for which a single-valued vector field $\underset{\sim}{\xi}$ can be found such that

$$\underset{\sim}{B}_1 = \mathrm{curl}(\underset{\sim}{\xi}\times\underset{\sim}{B}_o) \tag{3.11}$$

For these, the expression (3.8) for dc/dt is identical to that for d^2c/dt^2, [eq.(3.9], with $\underset{\sim}{\xi}$ replacing $\underset{\sim}{u}_1$. If eq.(3.11) holds, then $\underset{\sim}{\xi}$ can be interpreted as the displacement of magnetic field lines: A field line of the equilibrium field $\underset{\sim}{B}_o$, when shifted by an amount $\underset{\sim}{\xi}$, becomes a field line of the total field $\underset{\sim}{B} = \underset{\sim}{B}_o + \underset{\sim}{B}_1$. To demonstrate this, we represent the equilibrium field by

$$\underset{\sim}{B}_o = \nabla\alpha_o\times\nabla\beta_o \quad ,$$

and then show that, to first order in $\underset{\sim}{\xi}$, the total field satisfies

$$\underset{\sim}{B} = \nabla\alpha\times\nabla\beta,$$

where

$$\alpha(\underset{\sim}{x}+\underset{\sim}{\xi}) = \alpha_o(\underset{\sim}{x}), \beta(\underset{\sim}{x}+\underset{\sim}{\xi}) = \beta_o(\underset{\sim}{x}),$$

provided equation (3.11) holds. The computation proceeds as follows:

$$\begin{aligned}
\underset{\sim}{B} &= \nabla\alpha(\underset{\sim}{x})\times\nabla\beta(\underset{\sim}{x}) = \nabla\alpha_o(\underset{\sim}{x}-\underset{\sim}{\xi})\times\nabla\beta_o(\underset{\sim}{x}-\underset{\sim}{\xi}) \\
&= \nabla(\alpha_o(x)-\underset{\sim}{\xi}\cdot\nabla\alpha_o(\underset{\sim}{x}))\times\nabla(\beta_o(\underset{\sim}{x})-\underset{\sim}{\xi}\cdot\nabla\beta_o(\underset{\sim}{x})) \\
&= \nabla\alpha_o\times\nabla\beta_o-\nabla\alpha_o\times\nabla(\underset{\sim}{\xi}\cdot\nabla\beta_o)-\nabla(\underset{\sim}{\xi}\cdot\nabla\alpha_o)\times\nabla\beta_o \\
&= \nabla\alpha_o\times\beta_o+ \operatorname{curl}((\underset{\sim}{\xi}\cdot\nabla\beta)\nabla\alpha_o- (\underset{\sim}{\xi}\cdot\nabla\alpha_o)\nabla\beta_o) \\
&= \nabla\alpha_o\times\nabla\beta_o+ \operatorname{curl}(\underset{\sim}{\xi}\times(\nabla\alpha_o\times\nabla\beta_o)) \\
&= \underset{\sim}{B}_o + \operatorname{curl}(\underset{\sim}{\xi} \times \underset{\sim}{B}_o).
\end{aligned}$$

The fact that the linear circulation c is a linear function of time seems to contradict the conservation of the nonlinear circulation C. This apparent contradiction is resolved by observing that c does not approximate C in general. This is because c is computed by integrating along a field line of the unperturbed field $\underset{\sim}{B}_o$, while C is computed by integrating along a field line of the total field $\underset{\sim}{B}_o + \underset{\sim}{B}_1$. These two paths of integration are unrelated in general because given a closed field line of $\underset{\sim}{B}_o$ there need not be a closed field line of $\underset{\sim}{B}_o + \underset{\sim}{B}_1$ which approximates the former. To give an example, we consider the fields $\underset{\sim}{B}_o = \nabla\delta$, and

and $\underset{\sim}{B}_1 = \varepsilon \underset{\sim}{e}$, where θ is the angle about a straight axis, and $\underset{\sim}{e}$ is the unit vector along this axis. Then the field lines of $\underset{\sim}{B}_o$ are circles, but those of $\underset{\sim}{B}_o + \varepsilon \underset{\sim}{B}_1$ are helices which depart arbitrarily far from the circles even for arbitrarily small ε. In the light of these remarks, the fact that the linear circulation c is conserved if Eq. (3.11) holds agrees with the fact that $\underset{\sim}{\xi}$ then is a displacement vector by means of which a relation can be established between field lines of $\underset{\sim}{B}_o$ and field lines of $\underset{\sim}{B}_o$ and $\underset{\sim}{B}_1$ so that c indeed approximates C.

What can we now conclude? We cannot conclude that all plasmas are ill-behaved because experimental evidence shows the contrary. As already mentioned in Sec. 1, we can also not conclude that the MHD equations are no good. Thus, there are two possibilities: either we work with the nonlinear equations, or we try to find a better definition of stability within the linearized MHD model. Obviously, the second possibility is the only feasible one. This possibility is further justified by two facts: Firstly, confinement experiments are not intended to last forever; therefore, it is crucial to find out how much plasma and how much energy are lost due to instabilities within the intended duration of an experiment. Secondly, the MHD equations describe reality poorly in certain situations (e.g., if the magnetic field changes appreciably within one Larmor radius); therefore the spatial structure of an instability has to be known in order to decide whether it has to be taken seriously. For these reasons, a more detailed description of the solutions of the linearized MHD equations is called for.

4. Stability of a System with One Degree of Freedom

To introduce the various concepts commonly used in MHD stability theory, but whose significance and interrelations are not fully understood, we consider a completely tractable problem. The one-dimensional motion of a single particle subject to a conservative force is governed by the equation

$$\ddot{x} = F(x) \tag{4.1}$$

Here $x(t)$ is the position of the particle at time t, $F(x)$ is the force acting upon the particle at the position x, and dots denote time derivatives. The motion is uniquely determined for all times if both the position x and the velocity $\dot{x}$ are given initially. A particular position x_o is an equilibrium if the force is zero there, $F(x_o) = 0$. **The equilibrium is nonlinearly stable** if both the velocity and the deviation from the equilibrium position remain small forever provided they are small initially, or more precisely, if

$$|x(t_o)-x_o| \to 0 \quad \text{and} \quad |\dot{x}(t_o)| \to 0$$

imply that

$$|x(t)-x_o| \to 0 \text{ and } |\dot{x}(t)| \to 0 \text{ for all } t > t_o$$

In terms of the potential energy $U(x)$, defined by

$$F(x) = -U'(x), \qquad (4.2)$$

an equilibrium is characterized by

$$U'(x_o) = 0, \qquad (4.3)$$

and stability of this equilibrium is equivalent to the requirement that U has a local minimum at x_o. This statement is called "nonlinear energy principle." To indicate the proof, we note that equation (4.1) can be integrated once to yield the conservation of energy:

$$\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2+U\right) = 0 \qquad (4.4)$$

A further integration then yields the general solution in the implicit form

$$t-t_o = \int_{x(t_o)}^{x} \frac{dx}{\sqrt{2(E-U(x))}} \qquad (4.5)$$

where $E = \frac{1}{2}\dot{x}^2+U$ is the (constant) energy. From this it is seen by inspection that $x(t)$ and $\dot{x}(t)$ oscillate about x_o and 0 respectively, with amplitudes going to zero if $x(t_o) \to x_o$ and $\dot{x}(t_o) \to 0$, if U has a minimum at x_o, but that a solution with $x(t_o) = x_o$ and $\dot{x}(t_o) = \varepsilon$ departs away from x_o by an amount $O(1)$ even if $\varepsilon \to 0$ otherwise.

Next we discuss the linearized theory. When linearized about an equilibrium x_o, the equation of motion (4.1) becomes

$$\ddot{\xi} + U''(x_o)\xi = 0 \qquad (4.6)$$

where

$$\xi = x - x_o \tag{4.7}$$

is the displacement from the equilibrium. As before, we call the equilibrium "strictly linearly stable" if all solutions ξ are bounded as $t \to \infty$. A more optimistic definition is that of "exponential stability": no solution grows exponentially with time. A related, and still more optimistic definition is that of "σ stability": no solution grows faster than $\exp(\sigma t)$. To be precise,

$$\lim_{t \to \infty} \xi(t) \exp(-\alpha t) = 0$$

for $\alpha > \sigma$ in a σ stable equilibrium, and exponential stability is equivalent to σ stability for $\sigma = 0$. Obviously, σ_1 stability implies σ_2 stability if $\sigma_2 > \sigma_1$. Therefore, the concept of σ stability is more flexible than that of strict linear stability (or exponential stability): Rather than dividing all equilibria in just two classes labeled by "good" and "bad", it provides a continuous label (viz., the smallest σ for which the equilibrium is σ stable).

With the ansatz $\xi \propto \exp(i\omega t)$, equation (4.6) becomes

$$(-\omega^2 + U''(x_o))\xi = 0 \tag{4.8}$$

The set of those values of ω^2 for which there is a nontrivial solution is called the "spectrum" of the system. In the present simple case, the spectrum consists of the single real eigenvalue $\omega^2 = U''(x_o)$, and the general solution of equation (4.6) is

$$\xi = a \exp(i\omega t)+b \exp(-i\omega t) \qquad (4.9)$$

if $\omega^2 \neq 0$, but

$$\xi = a+bt \qquad (4.10)$$

in the exceptional case $\omega^2 = 0$. Here a and b are complex constants. To be physically meaningful, the real part of ξ has to be taken.

We now have to distinguish three cases:

1) U'' is positive at x_o. Then U has a local minimum, the spectrum is positive, and all solutions of the linearized equation of motion are osciallatory. The equilibrium is stable according to any of the above definitions.

2) U'' is negative at x_o. Then U has a local maximum, the spectrum contains a negative eigenvalue, and there are exponentially growing solutions. The equilibrium is both nonlinearly unstable and exponentially unstable. It is σ stable for $\sigma \geq \sqrt{-U''(r_o)}$, but σ unstable otherwise.

3) U'' vanishes at x_o. Then U may or may not have a local minimum depending on higher derivatives, the spectrum is non-negative but contains the origin, and there are solutions which grow linearly though not exponentially. The equilibrium is strictly linearly unstable, but exponentially stable, It may or may not be nonlinearly stable.

The following statements summarize what we have learned so far:

1) Exponential stability is equivalent to $U''(x_o) > 0$ ("linear energy principle") and σ stability is equivalent to $U''(x_o)+\sigma^2 \geq 0$ ("modified energy principle").
2) Strict linear stability implies nonlinear stability, nonlinear stability implies exponential stability, and exponential stability implies σ stability.
3) Strict linear stability is equivalent to $\omega^2>0$, exponential stability is equivalent to $\omega^2>0$, and σ stability is equivalent to $\omega^2+\sigma^2 \geq 0$.

To give a preliminary interpretation of the fact that most MHD equilibria are strictly linearly unstable we anticipate that what appears to be the exceptional case in the motion of a single particle (viz., $\omega^2 = 0$) is the common case in ideal MHD. We also anticipate that exponentially growing MHD instabilities always have a counterpart in the nonlinear theory, but that this need not be true for non-exponential instabilities.

5. Spectrum and Variational Principles in MHD

We now formulate generalizations to ideal MHD of the concepts of the last section. This is tentative to some extent because some of the statements which are trivially true for one particle have to be postulated for a plasma. The reason is, of course, that the theory of partial differential equations is less developed than that of ordinary differential equations.

Let us first formulate the nonlinear energy principle of ideal MHD. The analogy between the energy conservation laws (1.10) and (4.4.) suggests to postulate: A state is an equilibrium if the potential energy,

$$U = \iiint dt\left(\frac{1}{\gamma-1}p + \frac{1}{2} B^2\right), \qquad (5.1)$$

is stationery along all possible paths of the system through this state; an equilibrium is "energetically stable" if, in addition, U has a local minimum (again along all paths through the equilibrium). A restriction to paths of the system was unnecessary in our mechanical problem because there is only one possible path in a one-dimensional motion. If we omitted this restriction in the MHD problem, we would be left with the trivial equilibrium $p = 0$, $\underset{\sim}{B} = 0$. It is exactly this restriction which makes it difficult, if not impossible, to establish

the relation between energetic stability and some (reasonably defined) "nonlinear stability". The reason is that it does not seem to be possible to describe all paths through a given state without solving the equations of motion. A convenient way around this difficulty consists in replacing the above restriction by the requirement that certain constants of the motion (viz., those which are known, and which depend only on p and $\underset{\sim}{B}$, but not on $\underset{\sim}{u}$) are the same in all states to be considered. This gives rise to a different definition of stability, which we call "variational stability": An equilibrium is variationally stable if it minimizes the potential among all states with the same constants of motion. This statement is called "nonlinear energy principle".

The requirement of variational stability might be more restrictive than that of energetic stability because we are not certain to know all relevant constants of the motion, so that we might admit paths which are not accessible to the system. On the other hand, in the instability exhibited in connection with the circulation the system follows a path which does not pass through the equilibrium (the topology of the field lines is different), so that energetic stability might be an overly optimistic requirement. In summary, variational stability is a concept whose relation to the solutions

of the equations of motion is unknown. This is so despite the fact that it can be shown that the variational principle for equilibrium, using constants of the motion, is equivalent to the equilibrium differential equations (2.1-3), and despite the fact that there is a relation between the linear version of the energy principle and the solutions of the linearized equations of motion.

The nonlinear energy principle is tractable to some extent. Correspondingly, almost all the work which ever has been done in nonlinear MHD stability was based on the nonlinear energy principle.

Next we introduce the spectrum of linearized MHD. As before, we take the Fourier transform with respect to time of the perturbing quantities. Time derivatives in the linearized system (3.1-6) are then replaced by a factor $i\omega$. As a consequence, the perturbing density, pressure, and magnetic field can be expressed in terms of the velocity $\underset{\sim}{u}$, and we are left with an equation for $\underset{\sim}{u}$. Omitting subscripts we can write this equation as

$$\omega^2 \rho \underset{\sim}{u} = \underset{\sim}{F}\underset{\sim}{u} \qquad (5.2)$$

where $\underset{\sim}{F}$ is the matrix differential operator given by

$$\underset{\sim}{F}\underset{\sim}{u} = \underset{\sim}{B}\times \mathrm{curl}\,\mathrm{curl}(\underset{\sim}{u}\times\underset{\sim}{B}) + \mathrm{curl}(\underset{\sim}{u}\times\underset{\sim}{B})\times\underset{\sim}{J} - \nabla(\underset{\sim}{u}\cdot\nabla p+\gamma p\ \mathrm{div}\ \underset{\sim}{u}), \qquad (5.3)$$

and by the boundary condition $u_n = 0$. The perturbing field $\underset{\sim}{B}_1 \propto \text{curl}(\underset{\sim}{u} \times \underset{\sim}{B})$ then is automatically tangential to the boundary if the equilibrium field has this property.

It should be noted that linear growth of the circulation now has been eliminated because the perturbing magnetic field automatically satisfies Eq.(3.11) as a consequence of putting $\underset{\sim}{B}_1 \propto \exp(i\omega t)$. The motions in which the circulation grows correspond to $\omega = 0$, the only case in which elimination of $\underset{\sim}{B}_1$ and p_1 is a restriction, and which, therefore, is not completely covered by Eqs.(5.2-3).

It is mathematically convenient to imbed the solutions of equation (5.2) in the Hilbert space L^2 of complex-valued vector functions $\underset{\sim}{u}$ which satisfy the boundary condition and which are square integrable,

$$\iiint d\tau \, |\underset{\sim}{u}|^2 < \infty \, .$$

The scaler product in this space is defined as

$$(\underset{\sim}{u}_1, \underset{\sim}{u}_2) = \iiint d\tau \;\; \underset{\sim}{u}_1^* \cdot \underset{\sim}{u}_2 \, ,$$

where asterisks denote the complex conjugate. The spectrum then is defined as the set of those values of ω^2 for which the linear operator $(\underset{\sim}{F} - \omega^2 \rho)$ does not have a bounded inverse. According to the alternative theorem, those values of ω^2 for which equation (5.2) has a square-integrable solution belong to the spectrum. These are called point eigenvalues. However, there are, in general, values of ω^2 which belong to the spectrum but are not point eigenvalues. These are said to belong to the continuous spectrum. If

ω^2 belongs to the continuous spectrum, a sequence $\underset{\sim}{u}_k$ can be found for which $\lim_{k\to\infty}(\underset{\sim}{F}-\omega^2\rho)\underset{\sim}{u}_k = 0$, but which itself does not converge.

The operator $\underset{\sim}{F}$ has the property of being "hermitian", or "formally self adjoint". This means that

$$(\underset{\sim}{u}_1,\underset{\sim}{F}\underset{\sim}{u}_2) = (\underset{\sim}{F}\underset{\sim}{u}_1,\underset{\sim}{u}_2) \tag{5.4}$$

for any functions $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$ in the domain of $\underset{\sim}{F}$. (This domain is smaller than L^2 because not all square-integrable functions are twice differentiable). The property (5.4) follows from the identity

$$\underset{\sim}{u}_1^*\cdot\underset{\sim}{F}\underset{\sim}{u}_2-\underset{\sim}{F}\underset{\sim}{u}_1^*\cdot\underset{\sim}{u}_2 \tag{5.5}$$

$$= \text{div}[(\underset{\sim}{u}_2\times\underset{\sim}{B})\times\text{curl}(\underset{\sim}{u}_1^*\times\underset{\sim}{B})-(\underset{\sim}{u}_1^*\times B)\times\text{curl}(\underset{\sim}{u}_2\times\underset{\sim}{B})$$

$$+\gamma p(\underset{\sim}{u}_2 \text{ div } \underset{\sim}{u}_1^*-\underset{\sim}{u}_1^* \text{ div } \underset{\sim}{u}_2) + (\underset{\sim}{B}\cdot(\underset{\sim}{u}_1^*\times\underset{\sim}{u}_2))\underset{\sim}{J}]$$

upon integration, and using Gauss' theorem to show that the r.h.s. vanishes because of the boundary conditions. The identity (5.5) can be verified using the equilibrium equations (2.1-3). It should be noted that the operator $\underset{\sim}{F}$ is not selfadjoint in the strict sense because its adjoint $\underset{\sim}{F}^*$(defined by first finding, for each $\underset{\sim}{u}_1$, that $\hat{\underset{\sim}{u}}$ for which $(\underset{\sim}{u}_1, \underset{\sim}{F}\underset{\sim}{u}_2) = (\hat{\underset{\sim}{u}}_1,\underset{\sim}{u}_2)$ identically in $\underset{\sim}{u}_2$, and

then putting $\hat{u} = \underset{\sim}{F}^{*}\underset{\sim}{u}_{1}$) has a wider domain than $\underset{\sim}{F}$. Nevertheless, it can be shown that $\underset{\sim}{F}$ has a selfadjoint extension. This is crucial because the powerful theorems of functional analysis, on which some of the statements given below are based, would not apply otherwise. In what follows we will tacitly assume that we are dealing with that extension without introducing a new symbol for this.

As a consequence of the self-adjointedness of $\underset{\sim}{F}$, the spectrum is real. As we will see later, the structure of this spectrum is quite complicated: There are both point eigenvalues and continua, and the former frequently accumulate at the tips of the latter. The point eigenvalues are governed by a variational principle. In terms of the two quadratic functionals

$$W[\underset{\sim}{u}] = \frac{1}{2}(\underset{\sim}{u}, \underset{\sim}{F}\underset{\sim}{u}) \quad , \quad N[\underset{\sim}{u}] = \frac{1}{2}(\underset{\sim}{u}, \rho\underset{\sim}{u}) \tag{5.6}$$

This can be formulated as

$$\omega^{2} = W/N \quad , \; \delta\omega^{2} = 0, \tag{5.7}$$

(where the symbol δ denotes the variation with $\underset{\sim}{u}$), or equivalently

$$W-\omega^{2}N = 0, \quad \delta(W-\omega^{2}N) = 0 \tag{5.8}$$

The first version of this variational principle can be interpreted as requiring that W be stationery if $\underset{\sim}{u}$ is varied subject to the constraint N = 1. The second version arises from observing this constraint using a Lagrangian multiplier. To demonstrate the validity of the variational principle one simply shows that its Euler equation is equation(5.2).

A similar variational principle holds for the minimum (denoted by $\omega^2_{\min}$) of the spectrum:

$$\omega^2_{\min} = \inf \frac{W}{N} \quad , \tag{5.9}$$

or equivalently

$$\omega^2_{\min} = \max \left\{ \omega^2 \middle| W - \omega^2 N \geq 0 \right\} \tag{5.10}$$

If $\omega^2_{\min}$ is a point eigenvalue, this is, of course, a special case of the principle (5.7-8). However, it happens frequently that $\omega^2_{\min}$ belongs to the continuous spectrum; in this case the infimum of W/N (i.e., its largest lower bound) is not attained in L^2. $\omega^2_{\min}$ can be shown to be finite (i.e., $>-\infty$) as long as both $|\underset{\sim}{J}|$ and $1/\rho$ are bounded. On the other hand, $\omega^2_{\min}$ is never positive because $W[\underset{\sim}{u}] = 0$ for the trial functions $\underset{\sim}{u} = f(p)\underset{\sim}{B}+g(p)\underset{\sim}{J}$ with arbitrary f

and g. Thus, the origin is always a point eigenvalue with infinite multiplicity. It turns out that it also is the lower edge of a continuum.

We now call an equilibrium "σ stable" if the L^2 norm of the velocity cannot grow faster than $\exp(\sigma t)$ for large t,

$$\lim_{t\to\infty} ||\underset{\sim}{u}|| \exp(-\alpha t) = 0 \text{ for } \alpha>\sigma \tag{5.11}$$

where $||\underset{\sim}{u}|| = \sqrt{(\underset{\sim}{u},\underset{\sim}{u})}$. Correspondingly, we call it "exponentially stable" if it is σ stable for $\sigma = 0$. This definition is natural for two reasons: Firstly, it relates to the spectrum, and secondly, it relates to the kinetic energy of the system. As to the relation with the spectrum, it can be shown (using Laplace transform techniques and the spectral theorem for self-adjoint operators) that σ stability is equivalent to $\omega^2_{min} \geq - \sigma^2$. In other words, σ stability obtains if, and only if, the spectrum does not extend below $-\sigma^2$ (for this reason, $\sqrt{-\omega^2_{min}}$ is called the "maximum growth rate"). The variational principle (5.10) now implies that σ stability is equivalent to

$$W+\sigma^2 N \geq 0 \tag{5.12}$$

This is the "modified energy principle" of ideal MHD. For $\sigma^2 = 0$, this reduces to the ordinary energy principle, or better "linear energy principle": Exponential stability is equivalent to $W \geq 0$. The relation to the kinetic energy, defined by

$$K(t) = N[u(x,t)],$$

arises from the inequalities

$$\frac{1}{2}\rho_{\min} \|\underset{\sim}{u}\|^2 \leq K \leq \frac{1}{2}\rho_{\max} \|\underset{\sim}{u}\|^2$$

These imply that σ stability is equivalent to

$$\lim_{t\to\infty} K \exp(-2\alpha t) = 0 \text{ for } \alpha > \sigma \qquad (5.13)$$

We have not proved the relation between the spectrum and σ stability, as we have not proved the variational principle for the maximum growth rate. Instead, we will give an elementary proof of the equivalence of Eqs. (5.12) and (5.13) which is independent of the notion of the spectrum.

It would be tempting to define "strict energetic stability" as positivity of the spectrum, i.e., as $\omega^2_{\min} > 0$. However, we have already seen that $\omega^2_{\min} \leq 0$. Correspondingly,

it seems likely that perturbations can always be found for which K → ∞ as t → ∞ (in some non-exponential way). This has not been shown yet, and it does not follow from our investigation of "strict linear stability" because there a maximum norm was used as opposed to the present Hilbert norm (growth in the Hilbert norm would imply growth in the maximum norm, but the converse is not true).

For later reference, we now write down explicit expressions for the functional W. A simple integration by parts yields

$$W = \frac{1}{2} \iiint d\tau \Big\{ |\underset{\sim}{Q}|^2 + (\underset{\sim}{J} \times \underset{\sim}{u}^*) \cdot \underset{\sim}{Q} \tag{5.14}$$

$$+ (\underset{\sim}{u} \cdot \nabla p) \operatorname{div} \underset{\sim}{u}^* + \gamma p |\operatorname{div} \underset{\sim}{u}|^2 \Big\} ,$$

where

$$\underset{\sim}{Q} = \operatorname{curl}(\underset{\sim}{u} \times \underset{\sim}{B}) .$$

A particularly useful form of W is

$$W = \frac{1}{2} \iiint d\tau \Big\{ |\underset{\sim}{Q} + \frac{\underset{\sim}{J} \times \nabla p}{|\nabla p|^2} (\underset{\sim}{u} \cdot \nabla p)|^2 \tag{5.15}$$

$$+ A |\underset{\sim}{u} \cdot \nabla p|^2 + \gamma p |\operatorname{div} \underset{\sim}{u}|^2 \Big\} ,$$

where

$$A = -\frac{2(\underset{\sim}{J}\times\nabla p)\cdot(\underset{\sim}{B}\cdot\nabla)\nabla p}{|\nabla p|^4} \quad . \qquad (5.16)$$

For real $\underset{\sim}{u}$, this form is derived from Eq.(5.14) by using the identity

$$\begin{aligned}
&(\underset{\sim}{J}\times\underset{\sim}{u})\cdot\underset{\sim}{Q}+(\underset{\sim}{u}\cdot\nabla p)\ \mathrm{div}\ \underset{\sim}{u}\\
&= (\underset{\sim}{u}\cdot\nabla p)^2\ (A+\frac{\underset{\sim}{J}\times\nabla p|^2}{|\nabla p|^4})\\
&\quad + 2(u\cdot\nabla p)\frac{(\underset{\sim}{J}\times\nabla p)\cdot\underset{\sim}{Q}}{|\nabla p|^2}\\
&\quad + \mathrm{div}[\ \frac{(\underset{\sim}{u}\cdot\nabla p)(\underset{\sim}{u}\cdot(\underset{\sim}{J}\times\nabla p))}{|\nabla p|^2}\ \underset{\sim}{B}]\ ,
\end{aligned}$$

which, in turn, can be verified by making extensive use of the equilibrium equations.

6. Elementary Proof of the Modified Energy Principle

We wish to show that

$$\lim_{t\to\infty} K\, \mathrm{ep}(-2\alpha t) = 0 \tag{6.1}$$

for $\alpha > \sigma$, and for all solutions of the linearized MHD equations (3.1-6), is equivalent to

$$W_\sigma = W + \sigma^2 N \geq 0 \tag{6.2}$$

for all test functions in L^2. Our proof will be elementary in that no reference is made to the spectrum. It is also noteworthy that it will suffice to assume that the operator $\underset{\sim}{F}$ is hermitian (rather than strictly selfadjoint).

We relate the operator $\underset{\sim}{F}$ to the linearized equations (3.1-6) in two distinct ways, without taking their Fourier transform. Introducing a "displacement vector" $\underset{\sim}{\xi}$ by

$$\underset{\sim}{u} = \frac{\partial \underset{\sim}{\xi}}{\partial t} , \tag{6.3}$$

we can integrate Eqs. (3.4) and (3.6) to obtain

$$\underset{\sim}{B}_1 = \mathrm{curl}(\underset{\sim}{\xi}\times\underset{\sim}{B})+\hat{\underset{\sim}{B}} \tag{6.4}$$

$$p_1 = - \underset{\sim}{\xi}\cdot\nabla p-\gamma p \,\mathrm{div}\, \underset{\sim}{\xi} + \hat{p}, \tag{6.5}$$

where $\hat{\underset{\sim}{B}}(\underset{\sim}{x})$ and $\hat{p}(\underset{\sim}{x})$ are arbitrary functions (subject to $\mathrm{div}\, \hat{\underset{\sim}{B}} = 0$), and subscripts zero, as before, have been omitted in equilibrium quantities. Upon substitution of the expressions (6.4-5) into Eq. (3.1), this becomes

$$\rho \frac{\partial^2 \underset{\sim}{\xi}}{\partial t^2} + \underset{\sim}{F}\underset{\sim}{\xi} = -\nabla\hat{p} + (\text{curl } \underset{\sim}{\hat{B}})\times\underset{\sim}{B}+\underset{\sim}{J}\times\underset{\sim}{\hat{B}} \qquad (6.6)$$

Assuming now that the r.h.s. of Eq.(6.6) is zero, we restrict ourselves to the sub-class of solutions governed by the homogeneous equation

$$\rho \frac{\partial^2 \underset{\sim}{\xi}}{\partial t^2} + \underset{\sim}{F}\underset{\sim}{\xi} = 0 \ . \qquad (6.7)$$

It can be shown that these solutions are exactly those for which the system passes through its equilibrium state.

If, on the other hand, we differentiate Eq.(3.1) with respect to time, and then substitute Eqs.(3.4) and (3.6), we obtain

$$\frac{\partial^2 \underset{\sim}{u}}{\partial t^2} + \underset{\sim}{F}\underset{\sim}{u} = 0 \qquad (6.8)$$

However, only those solutions of Eq.(6.8) are associated with solutions of the original system (3.1-6) for which Eq.(3.1) holds initially, or more precisely, for which the initial acceleration

$$\underset{\sim}{a} = \rho\, \partial \underset{\sim}{u}/\partial t\big|_{t\,=\,0}$$

is such that

$$-\nabla\hat{p} + (\text{curl } \underset{\sim}{P}\times\underset{\sim}{B}+J\times\hat{B} = \underset{\sim}{a}$$

can be solved for $\hat{p}$ and $\underset{\sim}{\hat{B}}$.

Loosely speaking, we can say that the solutions of Eq. (6.8) contain those of the actual problem (3.1-6), and that the solutions of the actual problem contain those of Eq.(6.7). Therefore, Eq.(6.7) can be used to demonstrate necessity of the modified energy principle (which involves exhibiting one solution of the actual problem whose kinetic energy grows like exp(2σ) if W_σ can be made negative) and Eq.(6.8) can be used to demonstrate sufficiency (which involves showing that the kinetic energy grows slower than exp($2\sigma t$) for all solutions of the actual problem if $W_\sigma \geq 0$.

Necessity

Taking the scaler product of Eq. (6.7) with $\partial\underset{\sim}{\xi}/\partial t$

we find

$$\frac{d}{dt}(N[\partial\underset{\sim}{\xi}/\partial t] + W[\underset{\sim}{\xi}]) = 0$$

or upon integration,

$$N[\partial\underset{\sim}{\xi}/\partial t] + W[\xi] = E. \qquad (6.9)$$

This is the linearized version of the energy conservation (1.10). If, on the other hand, we take the scaler product of Eq.(6.7) with $\underset{\sim}{\xi}$, we compute

$$(\underset{\sim}{\xi},\rho\partial^2\underset{\sim}{\xi}/\partial t^2) = \frac{d}{dt}(\underset{\sim}{\xi},\rho\partial\underset{\sim}{\xi}/\partial t)-(\partial\underset{\sim}{\xi}/\partial t,\rho\partial\underset{\sim}{\xi}/\partial t)$$

$$= \frac{1}{2}\,\frac{d^2}{dt^2}(\underset{\sim}{\xi},\rho\underset{\sim}{\xi}) - (\partial\underset{\sim}{\xi}/\partial t,\rho\partial\underset{\sim}{\xi}/\partial t)$$

$$= \frac{d^2}{dt^2}\,N[\underset{\sim}{\xi}] - 2N[\partial\underset{\sim}{\xi}/\partial t]$$

to obtain the "virial equation"

$$\frac{d^2}{dt^2}\,N[\underset{\sim}{\xi}] - 2N[\partial\underset{\sim}{\xi}/\partial t]+2W[\underset{\sim}{\xi}] = 0 \qquad (6.10)$$

Upon elimination of W with the help of the energy conservation (6.9), this becomes

$$\frac{d^2}{dt^2}\,N[\underset{\sim}{\xi}] - 4N[\partial\underset{\sim}{\xi}/\partial t]+2E = 0 \qquad (6.11)$$

We now assume that $W_\sigma < 0$ for some trial function $\underset{\sim}{\eta}$, i.e.,

that

$$W[\underset{\sim}{\eta}] + \sigma^2 N[\underset{\sim}{\eta}] < 0 ,$$

and choose the initial data

$$\underset{\sim}{\xi}\Big|_{t = 0} = \underset{\sim}{\eta} , \quad \frac{\partial \underset{\sim}{\xi}}{\partial t}\Big|_{t = 0} = \sigma \underset{\sim}{\eta} \tag{6.12}$$

Eq.(6.9) then implies that E < 0, and Eq.(6.11) implies that

$$\frac{d^2}{dt^2} N[\xi] - 4N[\frac{\partial \underset{\sim}{\xi}}{\partial t}] > 0 . \tag{6.13}$$

Now, from Schwartz's inequality,

$$(\frac{d}{dt} N[\underset{\sim}{\xi}])^2 = (\frac{d}{dt} \frac{1}{2}(\underset{\sim}{\xi} , \rho \underset{\sim}{\xi}))^2 = (\frac{d}{dt} \frac{1}{2}(\sqrt{\rho}\, \underset{\sim}{\xi} , \sqrt{\rho}\, \underset{\sim}{\xi}))^2$$

$$= (\sqrt{\rho}\, \underset{\sim}{\xi} , \sqrt{\rho}\, \frac{\partial \underset{\sim}{\xi}}{\partial t})^2 \leq (\sqrt{\rho} \underset{\sim}{\xi}, \sqrt{\rho} \underset{\sim}{\xi})^2 \; (\sqrt{\rho} \frac{\partial \underset{\sim}{\xi}}{\partial t} , \sqrt{\rho} \frac{\partial \underset{\sim}{\xi}}{\partial t})^2$$

$$= 4N[\underset{\sim}{\xi}] \; N[\frac{\partial \underset{\sim}{\xi}}{\partial t}] , \tag{6.14}$$

Combining the two inequalities (6.13) and (6.14) we obtain

$$N[\underset{\sim}{\xi}] \frac{d^2}{dt^2} N[\underset{\sim}{\xi}] > (\frac{d}{dt} N[\underset{\sim}{\xi}])^2 \tag{6.15}$$

This is an ordinary differential inequality for $N[\underset{\sim}{\xi}]$, subject to the initial conditions

$$N[\underset{\sim}{\xi}] = N[\underset{\sim}{\eta}]$$

$$\frac{d}{dt}N[\underset{\sim}{\xi}] = 2\sigma N[\underset{\sim}{\eta}]. \qquad (6.16)$$

With the new variable

$$y = \log \frac{N[\underset{\sim}{\xi}]}{N[\underset{\sim}{\eta}]} \qquad (6.17)$$

Eq.(6.15) reduces to

$$\frac{d^2y}{dt^2} > 0 \ , \qquad (6.18)$$

and the initial conditions are

$$y = 0, \ \frac{dy}{dt} = 2\sigma \ . \qquad (6.19)$$

Integration then yields

$$\frac{dy}{dt} > 2\sigma, \quad y > 2\sigma t, \quad (\text{for } t > 0) \qquad (6.20)$$

or equivalently $\frac{d}{dt}N[\underset{\sim}{\xi}] > 2\sigma \ N[\underset{\sim}{\xi}]$,

$$N[\underset{\sim}{\xi}] > N[\underset{\sim}{\eta}] \exp 2\sigma t \quad . \qquad (6.21)$$

Using these two inequalities in Eq. (6.14), which, in terms of the kinetic energy $K = N[\partial \underset{\sim}{\xi}/\partial t]$, can be rewritten as

$$K > \left(\frac{d}{dt} N[\underset{\sim}{\xi}]\right)^2 / 4N[\underset{\sim}{\xi}], \tag{6.22}$$

we finally obtain

$$K(t) > K(0) \exp(2\sigma t) \tag{6.23}$$

Hence

$$\lim_{t\to\infty} K \exp(-2\sigma t) = K(0) > 0$$

Sufficiency

Taking the scaler product of Eq.(6.6) with $\partial \underset{\sim}{u}/\partial t$ we obtain

$$N[\partial \underset{\sim}{u}/\partial t] + W[\underset{\sim}{u}] = C \tag{6.24}$$

This is analogous to energy conservation, even though C is not the energy (it does not even have the dimension of an energy). If we assume that $W_\sigma \geq 0$, then

$$W[\underset{\sim}{u}] + \sigma^2 N[\underset{\sim}{u}] \geq 0. \tag{6.25}$$

After eliminating W with the help of Eq.(6.24) this inequality becomes

$$N[\partial \underset{\sim}{u}/\partial t] \leq C+\sigma^2 N[\underset{\sim}{u}] \quad . \tag{6.26}$$

On the other hand, the inequality (6.14), when written for $\underset{\sim}{u}$ rather than for $\underset{\sim}{\xi}$, is

$$(\frac{d}{dt}N[\underset{\sim}{u}])^2 \leq 4N[\underset{\sim}{u}]N[\partial \underset{\sim}{u}/\partial t] \; . \tag{6.27}$$

Combining the two inequalities (6.26) and (6.27), and observing that $N[\underset{\sim}{u}]$ is the kinetic energy, we obtain

$$\frac{dK}{dt} \leq 2\sqrt{\sigma^2 K^2 + CK} \tag{6.28}$$

Since $\sigma^2K^2+CK \leq (\sigma K+\frac{C}{2\sigma})^2$, this implies

$$\frac{dK}{dt} \leq 2\sigma K + C/\sigma,$$

and, upon integration,

$$K \leq (K(0)+\frac{C}{\sigma}t) \exp(2\sigma t) \quad .$$

Hence, for $\alpha > \sigma$,

$$\lim_{t \to \infty} K \exp(-2\alpha t) = 0$$

For $\sigma = 0$, Eq.(6.28) implies that K is bounded by a quadratic function of time, consistent with our conjecture that nonexponential growth of K is always possible.

7. Sufficient Stability Criteria

The advantage of the energy principle lies in the possibility of using inequalities. Therefore, rather than solving the Euler equation for the minimum of W_σ (which would be as difficult as solving the equations of motion), one can derive either necessary or sufficient conditions for stability. A necessary condition is optimistic in that an equilibrium may be unstable even if the condition is satisfied; a sufficient condition is pessimistic in that an equilibrium may be stable even if the condition is violated. Correspondingly, necessary conditions can be used only to show that an equilibrium is unstable, and sufficient conditions can be used only to show that an equilibrium is stable.

In the expression (5.15) for W, all terms are positive except the one involving the quantity A [Eq.(5.16)]. Hence $A \geq 0$, or explicitly,

$$(\underset{\sim}{J}\times\nabla p)\cdot(\underset{\sim}{B}\cdot\nabla)\nabla p \leq 0 \qquad (7.1)$$

is sufficient for exponential stability. In order to exhibit stabilizing equilibrium properties, we apply the condition (7.1) to various classes of equilibria.

Let us first assume that $\underset{\sim}{J}\cdot\underset{\sim}{B} = 0$. Then, from Eq.(2.1)

$$B^2\underset{\sim}{J}\times\nabla p = -|\nabla p|^2\underset{\sim}{B} \quad ,$$

and the condition (7.1) is equivalent to

$$\underset{\sim}{B}\cdot(\underset{\sim}{B}\cdot\nabla)\nabla p \geq 0 \qquad (7.2)$$

The identity

$$\underset{\sim}{B}\cdot(\underset{\sim}{B}\cdot\nabla)\nabla p+\nabla p\cdot(\underset{\sim}{B}\cdot\nabla)\underset{\sim}{B} = 0 \ ,$$

which follows from Eqs.(2.3-4), now implies that the condition (7.2) is equivalent to

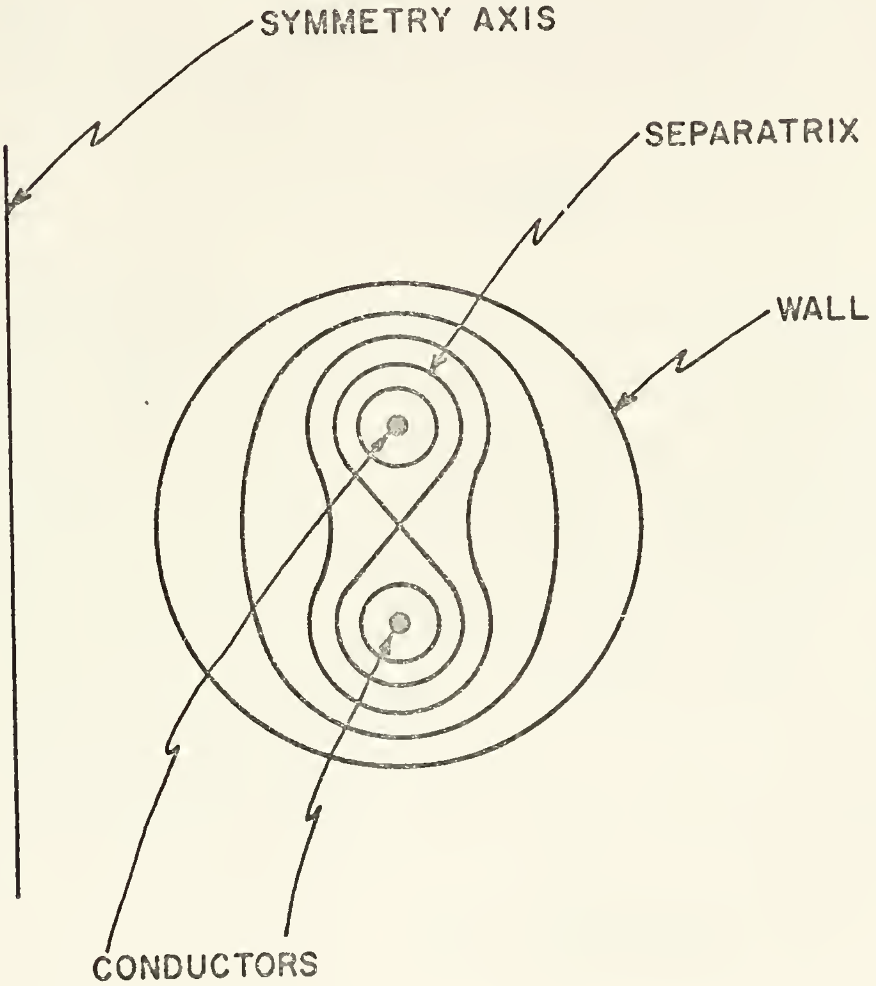
$$\underset{\sim}{\kappa}\cdot\nabla p \leq 0 \ , \qquad (7.3)$$

where $\underset{\sim}{\kappa}$ is the curvature of a magnetic field line,

$$\kappa = (\underset{\sim}{b}\cdot\nabla)\underset{\sim}{b} \ , \quad \underset{\sim}{b} = \underset{\sim}{B}/B \ .$$

Therefore, an equilibrium is exponentially stable if no current flows along the magnetic field lines, and if the field lines are nowhere concave towards increasing pressure. Obviously, this condition is never satisfied in a confined plasma. Nevertheless, it gives us a distinction between "good curvature" and "bad curvature". It can be shown that an equilibrium is stable if the curvature is good at a sufficiently large portion of each field line. The stability of axially symmetric "multipoles" is a consequence of this. In these equilibria the pressure has its maximum at the separatrix of the magnetic field. Inside the separatrix, the curvature is good everywhere;

outside the separatrix, it is good near the x point, but bad elsewhere. It can be shown that these equilibria are stable as long as the plasma does not extend too far away from the separatrix.



The curvature criterion (7.3) is still satisfied if the curvature is identically zero. This is the case if the magnetic field is unidirectional,

$$\underset{\sim}{B} = B(x,y)\underset{\sim}{e}_z \ .$$

Then

$$\underset{\sim}{J} = \mathrm{curl}\ \underset{\sim}{B} = \nabla B\times\underset{\sim}{e}_z \ .$$

Hence $\underset{\sim}{J}\cdot\underset{\sim}{B} = 0$, and the curvature criterion is valid. The pressure balance is

$$\nabla(p+\tfrac{1}{2}B^2) = 0 \quad .$$

An equilibrium of this kind can be constructed by choosing some domain in the (x,y) plane, then choosing some function B(x,y) which takes a constant value at the boundary, and finally putting $p = -\frac{1}{2}B^2+\mathrm{const.}$ The resulting equilibrium is exponentially stable. We call it "confined" if p has its minimum at the boundary. This is the case if B^2 has its maximum at the boundary, a property which is often called "magnetic well".

Writing the pressure balance (2.1) in the form

$$\nabla p = (\underset{\sim}{B}\cdot\nabla)\underset{\sim}{B}-\tfrac{1}{2}\nabla B^2$$

or equivalently

$$B^2\underset{\sim}{\kappa} + (\underset{\sim}{B}\cdot\nabla B)\underset{\sim}{b} = \nabla(p+\tfrac{1}{2}B^2),$$

we have

$$B^2\underset{\sim}{\kappa}\cdot\nabla p = \nabla p\cdot\nabla(p+\tfrac{1}{2}B^2) ,$$

and the criterion (7.3) becomes

$$\nabla p\cdot\nabla(p+\tfrac{1}{2}B^2) \leq 0 \qquad (7.4)$$

This implies that

$$\nabla p\cdot\nabla B^2 < 0 \qquad (7.5)$$

(unless $\nabla p = 0$), thus requiring that the magnetic field strength decreases in the direction in which the pressure increases. For a confined plasma in which the pressure decreases monotonically from its maximum to the wall, the condition (7.5) implies a magnetic well.

Good curvature and magnetic well are stabilizing properties as long as $\underset{\sim}{J}\cdot\underset{\sim}{B} = 0$. In order to show that this need not be so in general, we now give an example of a whole class of confined equilibria which are exponentially stable independently of the curvature of field lines and independently of the gradient of the magnetic field. These equilibria are characterized by $\partial/\partial z = 0$ (plane symmetry) and $J_z = 0$ (no current in the ignorable direction). Any solenoidal field with plane symmetry can be represented as

$$\underset{\sim}{B} = \underset{\sim}{e}_z\times\nabla\psi(x,y)+B_z(x,y)\underset{\sim}{e}_z .$$

Taking the curl of this, one finds

$$\underset{\sim}{J} = \underset{\sim}{e}_z \Delta\psi + \nabla B_z \times \underset{\sim}{e}_z,$$

and the pressure balance becomes

$$\Delta\psi\nabla\psi - B_z\nabla B_z + [e_z\cdot(\nabla\psi\times\nabla B)]e_z = \nabla p$$

This implies that both B and p are constant at the contours of constant ψ, and that

$$\Delta\psi = \frac{d}{d\psi}(p+\frac{1}{2}\,B_z^2).$$

Since $J_z = 0$ is equivalent to

$$\Delta\psi = 0, \tag{7.6}$$

the pressure balance in these equilibria simply reads

$$p+\frac{1}{2}B_z^2 = \text{const.} \tag{7.7}$$

An equilibrium can be constructed by first choosing an arbitrary domain in the (x,y) plane, then solving Eq.(7.6) with the boundary condition ψ = const, and finally assigning values of both p and B_z, consistently with Eq.(7.7), to each contour of constant ψ. In order that the solution of Eq.(7.6) is not just the trivial one (in which case we would deal with a unidirectional magnetic field,

as discussed before) we have to choose a multiply connected domain with different values of ψ at different disjoint parts of the boundary. This requires current carrying internal conductors in the z direction, so that we deal with a straight multipole. The stability of these equilibria follows from the fact that each of the three vectors $\underset{\sim}{J}$, ∇p, and $(\underset{\sim}{B}\cdot\nabla)\nabla p$ is in the (x,y) plane, so that $(\underset{\sim}{J}\times\nabla p)\cdot(\underset{\sim}{B}\cdot\nabla)\nabla p = 0$.

In a straight circular cylinder, the magnetic field is characterized by $B_r = 0, B_\theta(r), B_z(r)$, where (r,θ,z) are cylindrical coordinates. The current is then given by

$$J_r=0, \quad J_\theta = -\frac{d}{dr} B_z, \; J_z = \frac{1}{r}\frac{d}{dr} r\, B_\theta \quad ,$$

and the pressure balance is

$$\frac{d}{dr}(p+\tfrac{1}{2}B^2) + \frac{1}{r} B_\theta^2 = 0 \;.$$

The sufficient stability condition (7.1) now reduces to

$$J_z B_\theta \leq 0 \quad ,$$

or

$$\frac{d}{dr}(r^2 B_\theta^2) \leq 0 \tag{7.8}$$

The condition (7.8) cannot be satisfied in a simply connected domain unless $B_\theta \equiv 0$(unidirectional field) because $r^2 B_\theta^2 = 0$ for $r=0$, but $r^2 B_\theta^2 > 0$ for some $r > 0$.

Next we indicate how one can derive better (i.e., less restrictive) sufficient stability conditions. From Eq. (5.15),

$$W \geq \frac{1}{2} \iiint d\tau (|\underset{\sim}{F}|^2 + AX^2), \tag{7.9}$$

where

$$\underset{\sim}{F} = \underset{\sim}{Q} + \frac{\underset{\sim}{J} \times \nabla p}{|\nabla p|^2} X, \quad X = \underset{\sim}{\xi} \cdot \nabla p \quad .$$

Now, from Schwartz's inequality,

$$|\underset{\sim}{F}|^2 |\nabla p| \geq (\underset{\sim}{F} \cdot \nabla p)^2$$

Further,

$$\underset{\sim}{F} \cdot \nabla p = \underset{\sim}{Q} \cdot \nabla p = \nabla p \cdot \mathrm{curl}(\underset{\sim}{\xi} \times \underset{\sim}{B}) = \mathrm{div}[(\underset{\sim}{\xi} \times \underset{\sim}{B}) \times \nabla p] = \mathrm{div}(X\underset{\sim}{B}) = \underset{\sim}{B} \cdot \nabla X .$$

Hence

$$|\underset{\sim}{F}|^2 \geq \frac{(\underset{\sim}{B}\cdot\nabla X)^2}{|\nabla p|^2} \quad ,$$

and

$$W \geq \frac{1}{2}\iiint d\tau \left(\frac{(\underset{\sim}{B}\cdot\nabla X)^2}{|\nabla p|^2} + AX^2\right). \qquad (7.10)$$

This expression has two advantages: Firstly, it depends only on one component of the vector trial function $\underset{\sim}{\xi}$(viz., on X); secondly, only derivatives along magnetic field lines appear in the integrand.

Consider now equilibria either with an ignorable coordinate (like with axial symmetry) or with all magnetic field lines closed (constant rational rotation number). Then all physical quantities are periodic along magnetic field lines. In other words, if we introduce a coordinate s increasing along the field lines according to

$$ds = \frac{1}{q}\,\frac{d\ell}{B}\ , \quad q = \int \frac{d\ell}{B}$$

all physical quantities are periodic in s with period unity. Equation (7.10) can then be written as

$$W \geq \frac{1}{2}\iiint d\tau \left\langle \left(\frac{1}{q^2|\nabla p|^2}\ \frac{d^2X}{ds^2} + AX^2\right)\right\rangle ,$$

where

$$\langle \ldots \rangle = \int_0^1 ds \ldots$$

is an average along field lines. Obviously

$$\left\langle \frac{1}{q^2 |\nabla p|^2} \frac{d^2 X}{ds^2} + AX^2 \right\rangle \geq 0 \tag{7.11}$$

at every field line is sufficient for $W \geq 0$, and hence for exponential stability. The condition (7.11) is equivalent to the non-negativity of the spectrum of an eigenvalue problem (as the non-negativity of the MHD spectrum is equivalent to $W \geq 0$). This eigenvalue problem is given by

$$- \frac{d}{ds} \frac{1}{q^2 |\nabla p|^2} \frac{d}{ds} X + AX = \lambda g X \tag{7.12}$$

$$X(s+1) = X(s). \tag{7.13}$$

Eq. (7.12) is the Euler equation of the functional (7.11) with the normalization

$$\int_0^1 ds g X^2 = 1$$

The problem (7.12-13) is considerably easier to handle than the original problem not only in that it involves a second order ordinary differential equation rather than a sixth order system of partial differential equations, but

also in that it is of the Sturmian type: the spectrum is purely discrete, and there exists a lowest eigenvalue λ_o with a nonzero eigenfunction. $\lambda_o \geq 0$ at every field line is sufficient for stability. To determine λ_o requires numerical computations in general. Unlike for the original spectral problem, there are standard procedures for this.

It is also possible to derive lower bounds for λ_o analytically (if a lower bound is positive, λ_o is also positive, and the equilibrium is stable). The condition $\lambda_o \geq 0$ is certainly less restrictive than the previously discussed condition $A \geq 0$. In order to get a rough idea of how much better it can be at most we use the test function $X = 1$ in Eq.(7.11) to obtain $\langle A \rangle \geq 0$. This condition is necessary for $\lambda_o \geq 0$, and hence does not imply stability. It can be shown that $\langle A \rangle < 0$ at the magnetic axis whenever $\underset{\sim}{J} \neq 0$ there. Since this is always the case in Tokamak-like equilibria, the present sufficient condition is not good enough for investigating the stability of these equilibria.

Let us finally derive a sufficient condition for σ stability. From Schwartz' inequality,

$$|\underset{\sim}{\xi}|^2 |\nabla p|^2 \geq (\underset{\sim}{\xi} \cdot \nabla p)^2 = X^2$$

Hence

$$N \geq \frac{1}{2} \iiint d\tau \, \frac{1}{|\nabla p|^2} X^2 \quad ,$$

and, using Eq.(7.10), we find

$$W_\sigma \geq \frac{1}{2} \iiint d\tau \left(\frac{(B\cdot\nabla X)^2}{|\nabla p|^2} + AX^2 + \sigma^2 \rho \frac{X^2}{|\nabla p|^2} \right) .$$

For equilibria with periodicity along field lines,

$$\left\langle \frac{(B\cdot\nabla X)^2}{|\nabla p|^2} + AX^2 + \sigma^2 \rho \frac{X^2}{|\nabla p|^2} \right\rangle \geq 0 \qquad (7.14)$$

at every field line is sufficient for σ stability. This is equivalent to $\sigma^2 \geq -\lambda_o$, where λ_o is the lowest eigenvalue of the problem (7.12-13) with the weight function $g = \rho/|\nabla p|^2$. In other words, the maximum over all field lines of $-\lambda_o$ is an upper bound for the maximum growth rate for this choice of the weight function.

A simpler, but weaker, upper bound is obtained by dropping the first term in Eq.(7.14). This yields

$$\left(A + \frac{\sigma^2 \rho}{|\nabla p|^2}\right) X^2 \geq 0$$

as a sufficient condition for σ stability. This inequality holds for all X if, and only if,

$$A + \frac{\sigma^2 \rho}{|\nabla p|^2} \geq 0 ,$$

or

$$\sigma^2 \geq - \frac{A|\nabla p|^2}{\rho}$$

Hence the maximum of the r.h.s. over the whole domain is an upper bound for the maximum growth rate.

Sufficient criteria for exponential stability have never acquired much importance because they appear to be violated in most equilibria of practical interest. In accordance with this, the equilibria which we have shown to satisfy a sufficient criterion either are not confined in all three spatial directions because they are straight or have internal conductors (which have to be mechanically supported in practice). Fortunately, an equilibrium need not satisfy a sufficient criterion in order to be acceptable for plasma confinement, firstly because sufficient criteria are overly pessimistic, and secondly because instabilities with small growth rates or with large spatial variations are not always dangerous. Therefore, a normal mode analysis (i.e., an analysis of the spectrum together with the eigenfunctions) is required to indicate whether an equilibrium is acceptable. An intermediate solution of the problem is offered by evaluating sufficient criteria for σ stability.

8. Local Stability

Necessary criteria for exponential stability can be derived by considering special classes of trial functions in the energy principle. If a necessary criterion is violated, the equilibrium can be concluded to be unstable; but if it is satisfied, no conclusion can be drawn because not all possible trial functions were considered in its derivation. A specific necessary criterion arises if the subclass consists of "localized" trial functions, i.e., functions which are nonzero only in an arbitrarily small neighborhood of a pressure surface. The equilibrium is called "locally unstable" if this criterion is violated. It turns out that this criterion takes different forms depending on the topology of magnetic field lines. More precisely, the version which is valid for sheared equilibria does not reduce to the version which is valid for equilibria with all field lines closed in the limit of vanishing shear. This somewhat paradoxical fact indicates that the threshold for exponential stability is a discontinuous surface in parameter space, and, therefore, that exponential stability need not always be a very meaningful concept in practice. We give the derivation of the local stability criteria for circular cylinders.

These equilibria were described in Section 7. In order to simulate the toroidal structure of confined equilibria we identify any two planes z = const with

distance L, thus dealing with a topological torus. For simplicity, we put $L = 2\pi$. The rotation number is then given by

$$\mu(r) = B_\theta / rB_z . \tag{8.1}$$

We first consider the functional W_σ. Since θ and z are ignorable coordinates, this decomposes according to

$$W_\sigma[\underset{\sim}{\xi}] = \sum_{k,m} W_\sigma[\underset{\sim}{\xi}_{km}]$$

if we Fourier-decompose the trial function μ according to

$$\underset{\sim}{\xi} = \sum_{k,m} \underset{\sim}{\xi}_{km}(r) \exp i(m\theta+kz) \quad ,$$

$$\underset{\sim}{\xi}_{km}(r) = u_{km}(r)\underset{\sim}{e}_r + v_{km}(r)\underset{\sim}{e}_\theta + w_{km}(r)\underset{\sim}{e}_z$$

with integers m and k.

It follows that W_σ can be made negative if, and only if, it can be made negative for at least one Fourier-component of $\underset{\sim}{\xi}$. Therefore, we may consider one Fourier-component at a time. If we write W_σ explicitly in terms of the components of $\underset{\sim}{\xi}$, observing that derivatives with respect to θ and z are replaced by im and ik, we see that no derivatives of v and w appear. The minimization with respect to v and w thus can be performed algebraically (i.e., by completing

squares). Since the algebra, though straightforward, is extremely involved, we do not present it here, but merely write down the result (omitting subscripts k and m):

$$\min_{v,w} W_\sigma \propto \int dr[a_\sigma|(ru)'|^2+b_\sigma|u|^2] \quad , \qquad (8.2)$$

$$a_\sigma = \frac{AC}{rD} \quad ,$$

$$b_\sigma = A+2B_\theta(\frac{B_\theta}{r})'-4\frac{k^2B_\theta^2}{r^2}\,\frac{C-\sigma^2\rho\gamma p}{D}$$

$$+r[2\frac{kB_\theta}{r^2}(\frac{mBz}{r} - kB_\theta)\frac{C}{D}]' \quad ,$$

$$A = \sigma^2\rho+F^2$$

$$C = \sigma^2\rho(B^2+\gamma p)+\gamma pF^2$$

$$D = \sigma^4\rho^2+(\frac{m^2}{r^2} + k^2)C \quad ,$$

$$F = \frac{mB_\theta}{r}+kB_z = mB_z(\mu+\frac{k}{m}) \quad .$$

For exponential stability, we take $\sigma=0$. Then

$$A = F^2 \quad ,$$

$$C = \gamma pF^2 \quad ,$$

$$D = (\frac{m^2}{r^2} + k^2)\gamma pF^2 \quad ,$$

and

$$a_o = \frac{r^2F^2}{m^2+k^2r^2}$$

However, the limit as $\sigma \to 0$ of b_σ depends on whether $F \equiv 0$ or not. If $F \not\equiv 0$,

$$b_o = F^2+2B_\theta\left(\frac{B_\theta}{r}\right)' - \frac{4k^2B_\theta^2}{m^2+k^2r^2}$$

$$+r\left[2\frac{kB_\theta}{m^2+k^2r^2}\left(\frac{mB_z}{r} - kB_\theta\right)\right]'$$

(which is independent of γ); if $F \equiv 0$, there appears the additional term

$$\frac{4k^2B_\theta^2\gamma p}{(m^2+k^2r^2)(B^2+\gamma p)} \quad .$$

In the latter case, $a_o \equiv 0$. From Eq.(8.1) it is clear that $F=0$ at a particular surface if the perturbation is constant along the field lines there. If k and m are not both zero this is possible only if these field lines are closed. If the equilibrium has shear, then for a given pair (k,m) there may or may not be some isolated values of r at which $F=0$ ("mode rational surfaces"). If the shear is zero, then F either is identically zero or has no

zeroes at all, depending on whether k and m are such that the perturbation is constant along field lines or not (if μ is irrational, F is never zero, but can be made arbitrarily small by choosing k and m accordingly).

Let us now consider localized test functions u, i.e., we pick same surface $r=r_o$, put $r=r_o+\varepsilon x$ with a small parameter ε, and then assume that u is a smooth function of x (rather than of r), and that u=0 outside the interval $-1<x<1$. Since $u' = \frac{1}{\varepsilon}\frac{du}{dx}$, the first term in Eq.(8.2) is large compared to the second term in general. Since this first term is non-negative, we cannot make W negative in this way unless a_o (and hence also F) is zero at r_o. We thus pick a mode rational surface if the equilibrium has shear. If the shear is zero, the present procedure does not yield a stability condition unless all field lines are closed. In the latter case we have to choose k and m such that $F\equiv 0$.

First considering equilibria with shear, we expand W_o to lowest order in ε to obtain

$$W_o \propto \frac{1}{2}\, a_o''(r_o) \int_{-1}^{1} dx \left|x\,\frac{du}{dx}\right|^2 + b_o(r_o) \int_{-1}^{1} dx\,|u|^2$$

If this can be made negative, the exact W_o can also be made negative by making ε sufficiently small. Let us first minimize the first term keeping the second term fixed. By Schwartz's inequal

$$\int dx \left|x\,\frac{du}{dx}\right|^2 \int dx\,|u|^2 \geq \left|\int dx\ \frac{du^*}{dx}\,ux\right|^2 .$$

Denoting the r.h.s. by $|I|^2$, we further have $|I|^2 \geq (\mathrm{Re}\ I)^2$.

But

$$\mathrm{Re}\ I = \frac{1}{2}(I+I^*) = \frac{1}{2}\int dx\ x\left(\frac{du^*}{dx}u+\frac{du}{dx}u^*\right) = \frac{1}{2}\int x\ \frac{d}{dx}|u|^2 = -\frac{1}{2}\int dx|u|^2$$

Hence

$$\int dx\left|x\frac{du}{dx}\right|^2 \geq \frac{1}{4}\int dx|u|^2 \quad ,$$

and

$$W_o \propto \left(\frac{1}{8}\ a_o''(r_o)+b_o(r_o)\right)\int dx|u|^2$$

Now using $F(r_o) = 0$, we find that

$$\frac{1}{8}\ a_o''(r_o) + b_o(r_o) \propto p'+\frac{r}{8}\ B_z^2\left(\frac{\mu'}{\mu}\right)^2\bigg|_{r=r_o}$$

This expression depends on k and m only through r_o. If it is negative for any pair (k,m), or equivalently, at any rational surface r_o, W_o can be made negative, and the equilibrium is unstable. Since the rational surfaces are dense, it is negative at some rational surface if it is negative anywhere. Hence

$$p' + \frac{r}{8}\ B_z^2(\frac{\mu'}{\mu}\ |^2 \geq 0 \qquad (8.3)$$

is necessary for exponential stability, provided $\mu' \not\equiv 0$.

This is Suydam's condition.

Next we consider equilibria with all field lines closed. As already noted we then have to take $F \equiv 0$. We thus have

$$W_o = \int dr\, b_o\, |u|^2$$

where b_o includes the extra term depending on γ. Using $F \equiv 0$ (and the equilibrium equations) we find

$$b_o \propto \left(p' + \frac{2\gamma p}{B^2+\gamma p} \frac{B_\theta^2}{r}\right)\mu^2$$

Obviously,

$$p' + \frac{2\gamma p}{B^2+\gamma p} \frac{B_\theta^2}{r} \geq 0 \qquad (8.4)$$

is necessary for stability provided $\mu \neq 0$.

In the limit of zero shear ($\mu' \to 0$), Suydam's criterion (8.3) becomes $p' \geq 0$, and thus is more restrictive than the criterion (8.4). Nevertheless, it cannot be concluded that the criterion (8.4) is weaker than Suydam's criterion, and therefore redundant, because Suydam's criterion is not valid whenever the criterion (8.4) is valid. The correct conclusion is that the stability threshold is a discontinuous function of μ' at $\mu' = 0$. For instance, Suydam's condition implies that a confined plasma ($p' < 0$) is locally unstable if the shear is too small, whereas the criterion (8.4) implies that it can be locally stable if the shear is zero.

The local criteria (8.3) and (8.4) have been generalized to arbitrary equilibria with shear and with closed field lines, respectively. Due to the absence of symmetries, the derivation of these general local criteria is extremely involved, and cannot be given here. Since the explicit form of these criteria is not very intuitive either (i.e., stabilizing and destabilizing effects are not directly visible), we do not even write them down. However, it should be remarked that a large body of the literature on MHD stability is concerned with the evaluation of these local criteria for specific equilibria. For this reason, it is of extreme importance to have some information about the nature of the instabilities that occur if the local criterion is violated. For circular cylinders, this information is derived in Sec. 9.

9. Normal Modes

We derive properties of the unstable spectrum in circular cylinders. We recall that normal modes are obtained by making the functional W_σ stationery. For circular cylinders, this functional is given by Eq.(8.2). Making it stationery leads to the Euler equation

$$\frac{d}{dr} a_\sigma \frac{d}{dr} ru = b_\sigma u \tag{9.1}$$

With the boundary conditions u=0 at r=0(regularity of the vector $\underset{\sim}{\xi}$) and at r=R (perfectly conducting rigid wall), Eq.(9.1) constitutes an eigenvalue problem with eigenvalue parameter σ^2.

This problem is non-standard not only in that the coefficients a_σ and b_σ depend on σ^2 in a complicated manner, but also in that the relevant function space is not simply that of square integrable functions u(r), but rather that defined by the square-integrability of the vector $\underset{\sim}{\xi}$ (the underlying norm involves derivatives of u, and thus is a Sobolev norm). Therefore, the well-developed theory of the spectrum of linear second order ordinary differential operators is not applicable. For instance, there does not seem to be a simple way of showing directly from Eq.(9.1) that the spectrum is real (a fact which we know to be true from more general considerations).

An obvious way to find eigenvalues is the "shooting method": One solves Eq.(9.1) as an initial value problem with initial conditions at r=0. The value of u at r=R then is a function of σ^2, say $f(\sigma^2)$. Those values of σ^2 for which f=0 are the eigenvalues of the problem. The eigenvalues are isolated as long as the function f is well-behaved, and this is the case as long as Eq.(9.1) is non-singular, i.e. $a_\sigma/b_\sigma \neq 0$. This is the case for $\sigma^2>0$. Hence the unstable spectrum is discrete, and continua may occur only at the stable side. In particular, unstable eigenvalues cannot accumulate anywhere except at the origin. However, it should be noted that this conclusion holds only for fixed mode numbers k and m. If we form the entire spectrum by taking the union of all the subspectra pertaining to fixed k and m, we may obtain unstable accumulation points.

We now derive a monotonicity property of the unstable eigenvalues: σ^2 is a monotonically decreasing function of the number of zeroes of the associated eigenfunction. This number is called "radial mode number", and we denote it by n. Since monotonicity in n of the eigenvalues is a well-known property of Sturmian problems[(viz., eigenvalue problems of the form $(fu')'+gu = \lambda u$)], it is sometimes referred to as the "Sturmian property", even though the problem (9.1) is not of the Sturmian type. In order to demonstrate the Sturmian property, we denote the solution of the above initial value

problem by $u(r,\sigma^2)$, and the n-th zero of this by $r_n(\sigma^2)$. Since the eigenvalues are given by $r_n(\sigma^2)=R$, it suffices to show that r_n is a monotonic function of σ^2 for $\sigma^2>0$. If we assume analyticity of the function $r_n(\sigma^2)$, this follows from the reality of the spectrum through the following well-known theorem: If an analytical function $f(Z)$ is real on some curve in the complex Z plane, and if f has a local extremum along this curve at some point Z_o, then f is also real on at least one second curve through the same point Z_o. This theorem is easily proved by considering the power series of f about the point Z_o. In our present application, the function $r_n(\sigma^2)$ takes the place of $f(Z)$, and the curve on which f is real is the real axis. If we assume that r_n is non-monotonic, it follows that r_n must have a local extremum. The theorem then implies that r_n is real for complex values of σ^2. Hence the equation $r_n(\sigma^2)=R$ has complex solutions σ^2 for real R. Since this contradicts the fact that the spectrum is real, the function $r_n(\sigma^2)$is monotonic.

The Sturmian property has several practical consequences: Firstly, it implies that the n=0 mode which is the mode best described by ideal MHD because it has the smallest possible spatial variation, is also the most important one because it has the largest growth rate. Secondly, it implies that the growth rates decrease if the wall radius is decreased, an effect which is called "wall stabilization". Thirdly, it

implies that an equilibrium is σ stable (for the given mode numbers m and k) if, and only if, $r_1(\sigma^2) \geq R$, or equivalently, if no solution of the Euler equation (9.1) has more than one zero in the open interval $0<r<R$.

We now ask the question what happens as we let $n\to\infty$. The Sturmian property and the discreteness of the unstable spectrum imply that there are two possibilities: Either the eigenvalues accumulate at the origin, or there are only finitely many unstable eigenvalues (in which case the counting breaks down as the origin is crossed because Eq.(9.1) then is singular).

As one would anticipate, accumulation takes place in equilibria with shear if Suydam's criterion is violated at a mode rational surface. To demonstrate this it suffices to show that the number of zeroes of the solutions $u(r,\sigma^2)$ of Eq.(9.1) tends to infinity as σ^2 tends to zero (if Suydam's criterion is violated). Eq.(9.1) is singular at the mode rational surface $r=r_o$ if $\sigma^2=0$. Near this surface, it takes the form

$$(x^2u')' = cu, \tag{9.2}$$

where

$$x = r-r_o \quad ,$$

$$c = \frac{2b_o}{d^2a_o/dr^2}\Bigg|_{r=r_o} \quad .$$

The behavior of the solutions of Eq.(9.1) near the singularity is given by the solutions of Eq.(9.2). The latter are

$$u \propto \frac{\exp(\beta \log|x|)}{\sqrt{|x|}} \qquad (9.3)$$

where

$$\beta = \pm\sqrt{c+\frac{1}{4}} \quad .$$

The function (9.3) has infinitely many zeroes near x=0 if, and only if,

$$c + \frac{1}{4} < 0$$

This is the case if Suydam's condition is violated.

In closed line equilibria, Eq.(9.1) is singular for $\sigma^2=0$ only if k and m are such that $k+m\mu\equiv 0$ (resonant modes). Hence no accumulation of unstable eigenvalues takes place for non-resonant modes. For resonant modes, as one would again anticipate, the origin is an accumulation point of unstable eigenvalues if, and only if, the local stability criterion (8.4) is violated. It again suffices to show that the number of zeroes of $u(r,\sigma^2)$ becomes large as σ^2 becomes small. For $k+m\mu=0$, and small σ^2, Eq.(9.1) becomes

$$\sigma^2 \frac{d}{dr} \frac{\rho r}{m^2+k^2r^2} \frac{d}{dr} ru = b_o u. \qquad (9.4)$$

Since the coefficient of u" is small, (and positive) the solutions oscillate rapidly (and thus have many zeroes) wherever $b_o<0$. This is the case wherever the local stability condition (8.4) is violated.

How can we now understand the fact that there are two distinct local stability criteria for $\mu'\neq 0$ and for $\mu'=0$? This fact seems even more paradoxical if we observe that, by the same argument by which we concluded that the unstable spectrum is discrete, each unstable eigenvalue depends continuously on equilibrium parameters. The answer is that this continuity is non-uniform in the mode number n. Thus, the limits $n\to\infty$ and $\mu'\to 0$ are not interchangeable: Taking first $n\to\infty$, and then $\mu'\to 0$, we find $\omega^2\to -0$ if $p'<0$, but taking first $\mu'\to 0$, and then $n\to\infty$, we find $\omega^2\to -0$ if the criterion (8.4) is violated. Similar non-uniformities can be found for other limits (e.g., $m\to\infty, k\to\infty, |p'|\to\infty$, etc.).

The following statements summarize what we have learned about the unstable spectrum:

1) The unstable spectrum consists of point eigenvalues, which depend continuously on the equilibrium.

2) Each unstable point eigenvalues is uniquely characterized by the three mode numbers k,m, and n.

3) For fixed k and m, the growth rate decreases as n increases, and tends to zero as $n\to\infty$(if the $n\to\infty$ modes are unstable at all).

4) For fixed k and m, the stability of the $n\to\infty$ modes depends on the local stability criterion. If $\mu' \neq 0$, these modes are unstable if, and only if, Suydam's criterion is violated at a mode-rational surface; if μ is a rational constant, they are unstable if, and only if, the criterion (8.4) is violated anywhere in the plasma.

From a practical point of view, only those unstable modes are important whose mode numbers are small ("gross modes"). The $m=\pm 1$, $k=\pm 1$, $n=0$ modes, for instance, are of particular importance in equilibria with shear because they are frequently observed in experiments. From the above statements it is clear that these modes are unstable if Suydam's criterion is violated at a $\mu=\pm 1$ surface. More generally, the $n=0$ mode is unstable for any given k and m if Suydam's criterion is violated at a mode rational surface. Suydam's criterion thus is a condition for gross stability, even though it is derived by considering localized test functions, but only at the low-order rational surfaces. If there are no such surfaces, as it is the case in equilibria with small shear, Suydam's criterion is not related to gross modes. As to the relevance to gross modes of the criterion (8.4) for closed line equilibria, there is no restriction to particular surfaces because all surfaces are mode rational if $k/m+\mu=0$. Hence the criterion is relevant as long as μ is a low order rational number. By continuity, it is still relevant if μ is close to a low order rational number (e.g., small μ and low shear) Needless to say that these practical conclusions

should also apply to the generalizations to arbitrary equilibria of the local criteria (8.3) and (8.4).

As a final remark we stress that gross MHD stability of experimental plasmas is a quantitative numerical problem. The local criteria are only necessary, but not sufficient for gross stability, and thus yield only a bound for the threshold. The actual threshold, unlike the local criteria, is not related to a singularity of the equations, and thus cannot be determined analytically. This is true even more for the growth rates of gross modes. And if there is anything in the linearized MHD theory which is relevant to plasma confinement, then these are the growth rates of gross modes.

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